Definition: Radians

When given a circle, a radian measures the number of radii that you travel around the circumference of a circle.

We note that the formula for a circumference is $C = 2\pi r$ and we note that the number of radii needed to travel around a circle can be found by calculating $\frac{C}{r} = 2\pi$.

This gives us the formula for converting between degrees and radians as follows. Since 2π corresponds to 360^0 we can reduce this to get a ratio of $\pi : 180^0$

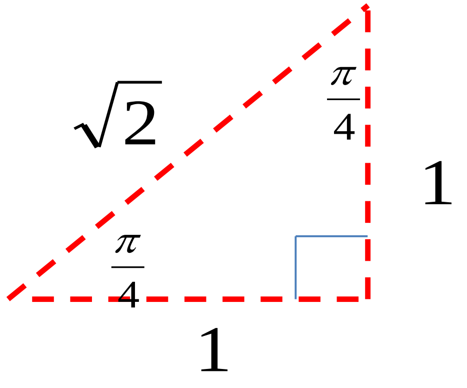
Let θ be a measure in degrees, and let α be the radian measure corresponding to θ. Then we have:

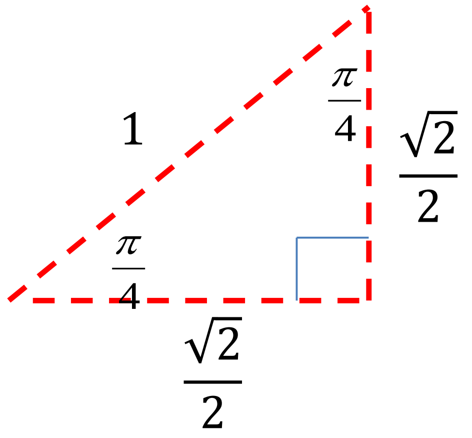
$$\frac{\alpha}{\pi} \times 180 = \theta \quad \text{or} \quad \frac{\theta}{180} \times \pi = \alpha$$

Definition: Special Triangles

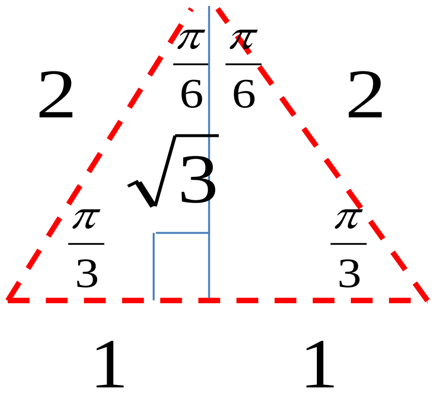
There are two triangles that are constructed from special cases (called special triangles) from what we understand from isosceles and equilateral triangles.

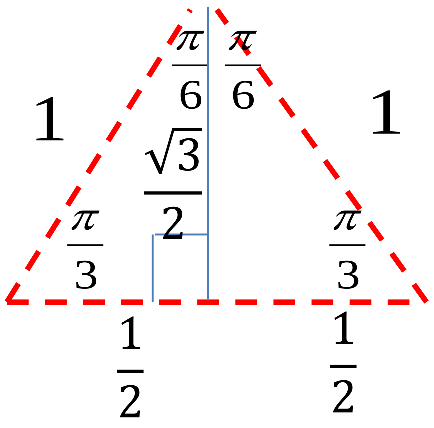
Consider an isosceles triangle with legs of length 1, we would get the following triangle:





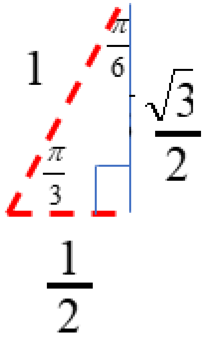
Consider an equilateral triangle with legs of length 2, we would get the following triangle:

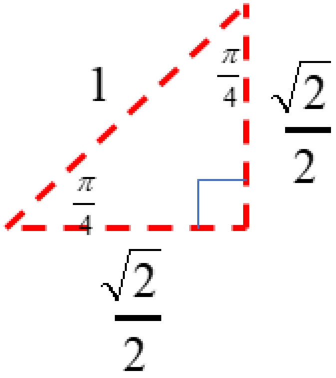


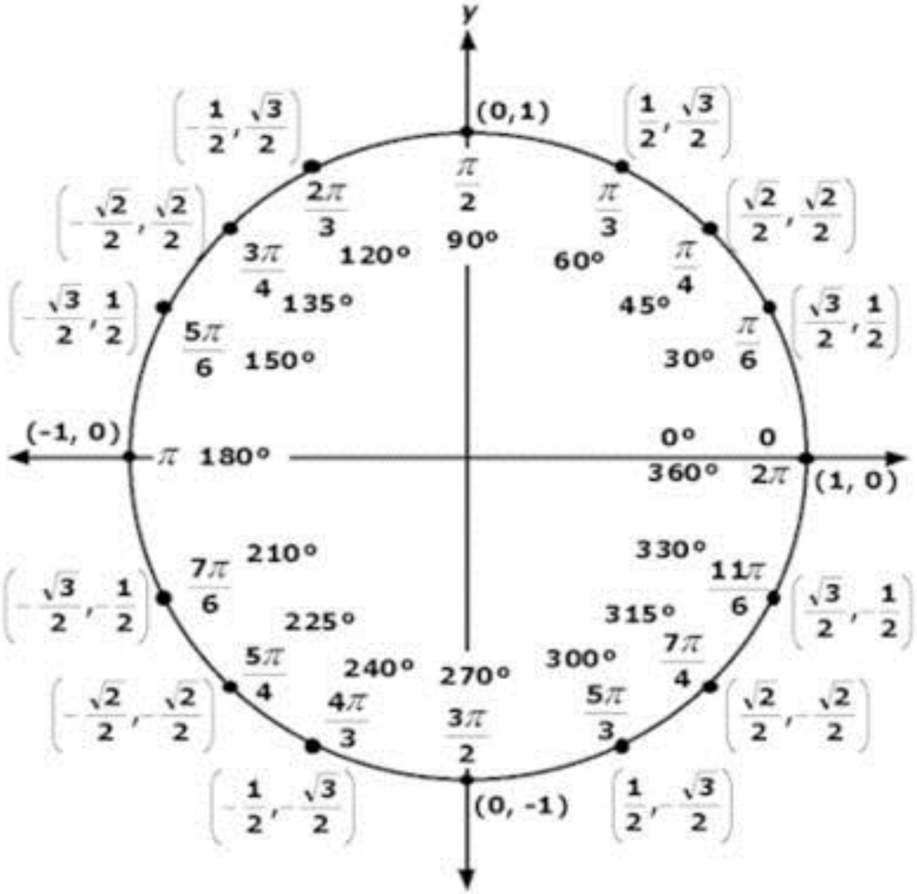


Definition: Unit Circle and Trigonometric Ratios

If we take our special triangles and force the hypotenuse to be 1, we can then plot these triangles onto a circle with radius 1. This gives us what is known as the unit circle:







Given a circle of radius 1 and a point on the circle (x, y) and an angle starting from the positive x axis going counter clockwise given by θ, we define the following <u>trigonometric ratios</u> using the **<u>unit circle above</u>**:

$$\sin(\theta) = y \qquad \cos(\theta) = x \qquad \tan(\theta) = \frac{y}{x} \qquad \csc(\theta) = \frac{1}{y} \qquad \sec(\theta) = \frac{1}{x} \qquad \cot(\theta) = \frac{x}{y}$$

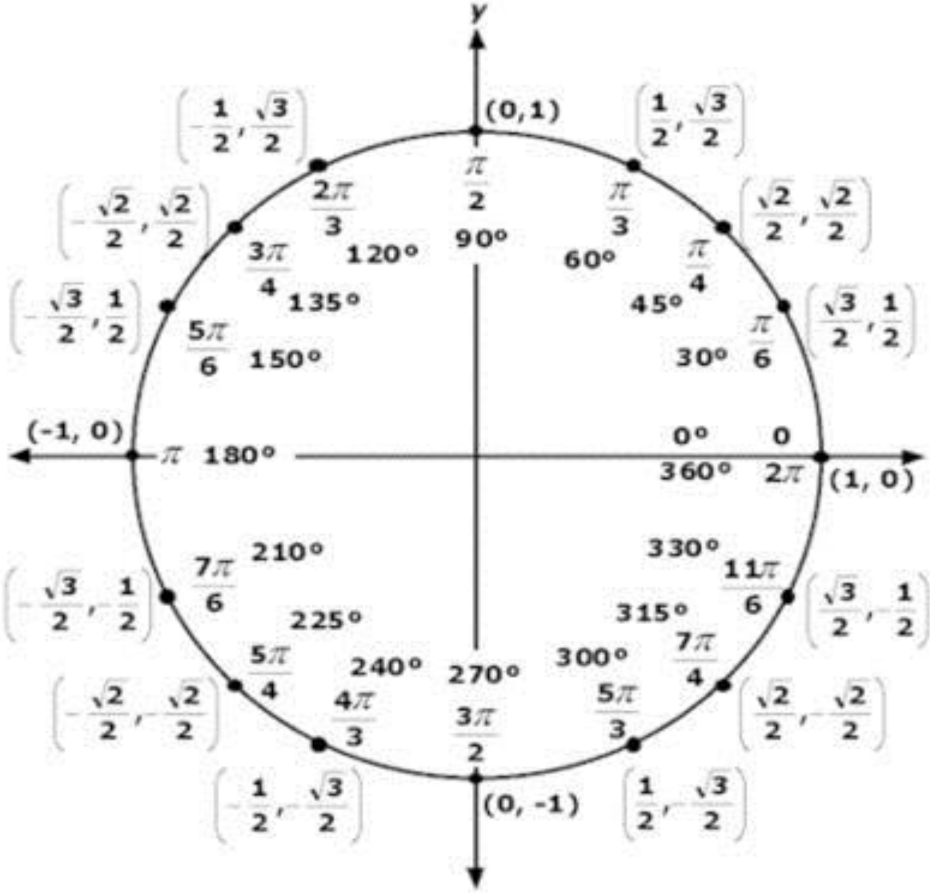
Examples: Using the Unit Circle

Example 1:

Determine the quadrant(s) where $Cos(\theta)$ will be positive:

Solution:

We require $\cos(\theta) = x$ to be positive.



This happens in quadrant I and IV.

Examples: Using the Unit Circle

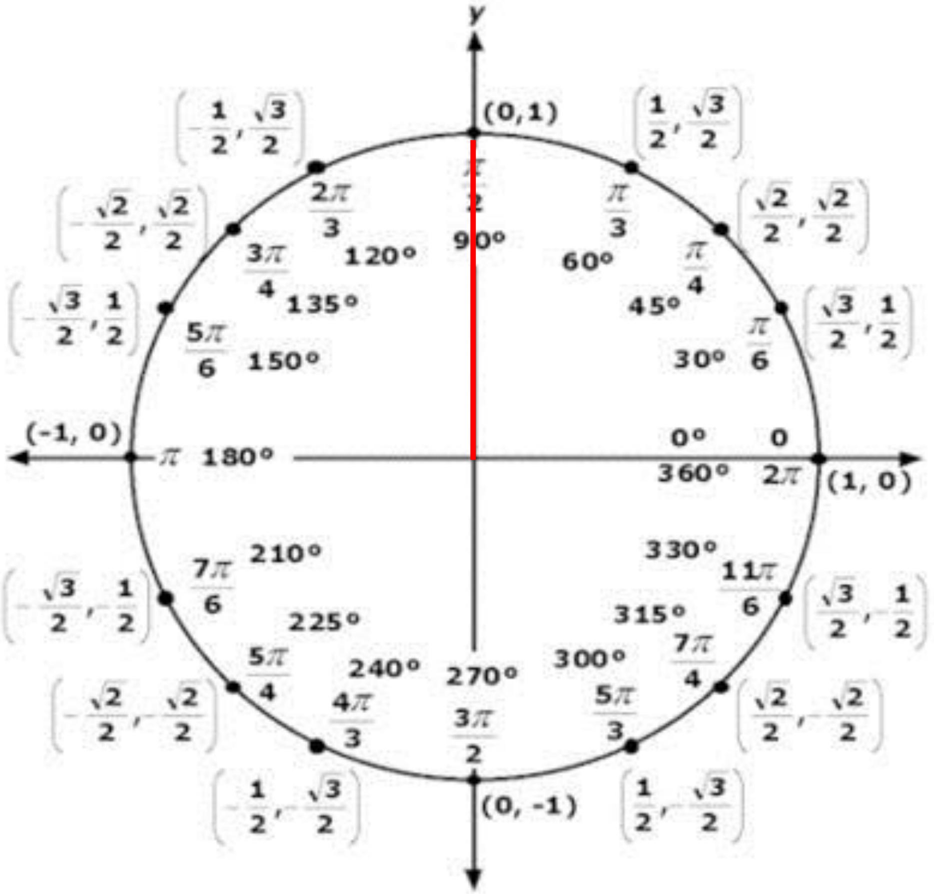
Example 2:

Determine when $\sin(\theta) = 1$

Solution:

We require $\sin(\theta) = y = 1$.

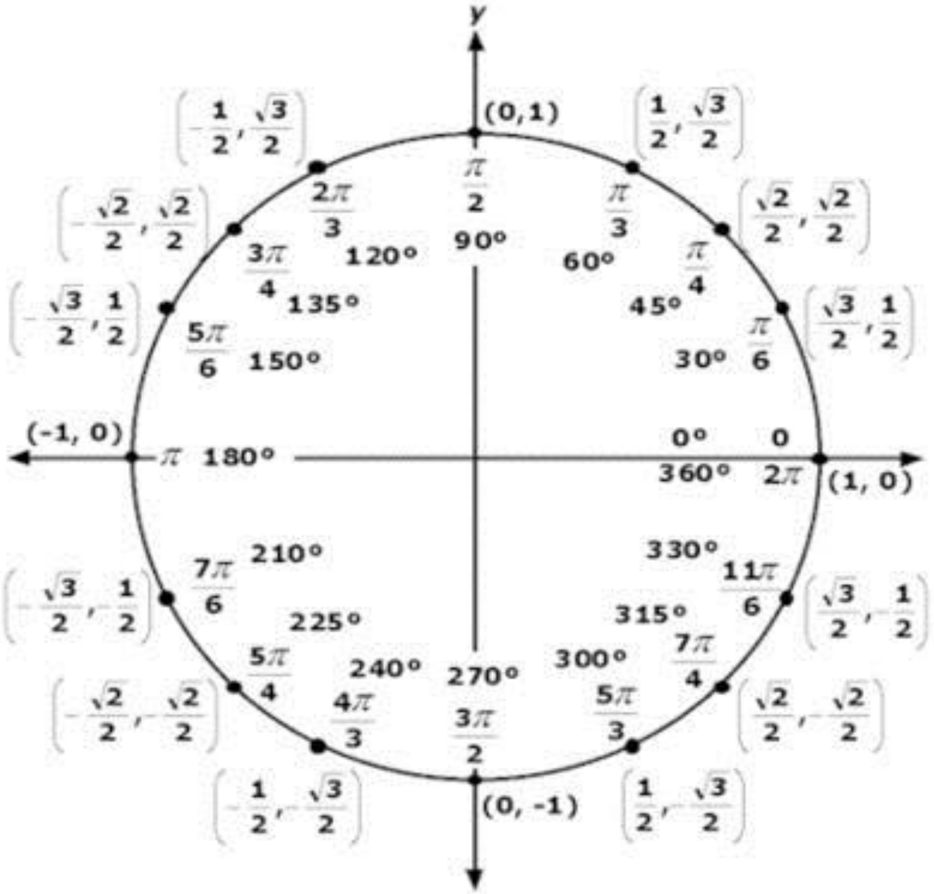
Looking at the graph, this only happens at $\theta = \frac{\pi}{2}$



Examples: Using the Unit Circle

Example 3:

Determine when $\sin(\theta)$ when $\tan(\theta) = \frac{3}{4}$ and $\pi \leq \theta \leq 2\pi$



Solution:

We note that $\tan(\theta) = \frac{y}{x}$ which leads us to believe that $x = 4$ and $y = 3$, however, we have a domain of $\pi \leq \theta \leq 2\pi$, which means the only way for $\tan(\theta)$ to be positive is if $x = -4$ and $y = -3$.

Secondly, since we need to be on the unit circle, if we have $x = -4$ and $y = -3$, this would give us a radius of $(-4)^2 + (-3)^2 = r^2$ which gives $r = 5$. Since we need a unit circle, we actually need to scale our x and y by $1/5$ so that the radius will be 1. This means we get $x = -\frac{4}{5}, y = -\frac{3}{5}$ which means $\sin(\theta) = y = -\frac{3}{5}$

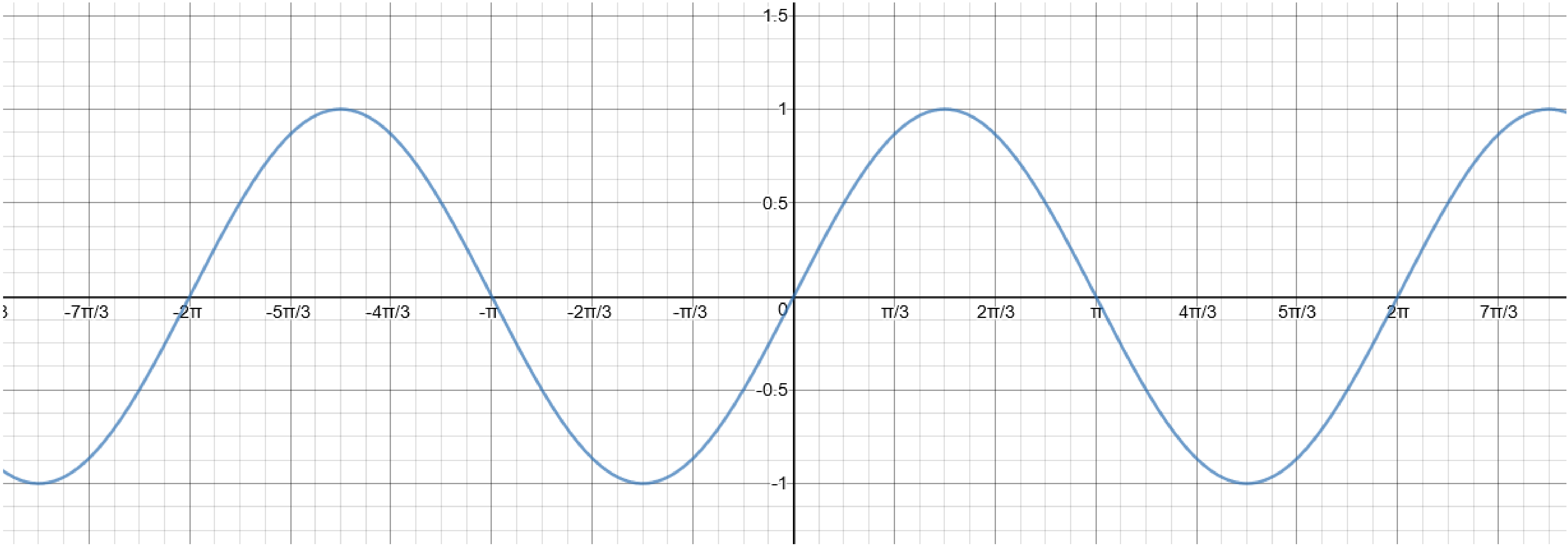
Definition: Trigonometric Parent Functions

Using the unit circle, we can construct a table of values for each trigonometric function. We can then use this information to construct the corresponding graphs of the trigonometric functions that relates ratios (r) with the angle (θ):

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin(\theta) = y$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos(\theta) = x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\tan(\theta) = \frac{y}{x}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0
$\csc(\theta) = \frac{1}{y}$	und	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	und	-2	$-\sqrt{2}$	$\frac{-2\sqrt{3}}{3}$	-1	$\frac{-2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	und
$\sec(\theta) = \frac{1}{x}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	und	-2	$-\sqrt{2}$	$\frac{-2\sqrt{3}}{3}$	-1	$\frac{-2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	und	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
$\cot(\theta) = \frac{x}{y}$	und	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$\frac{-\sqrt{3}}{3}$	-1	$-\sqrt{3}$	und	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$\frac{-\sqrt{3}}{3}$	-1	$-\sqrt{3}$	und

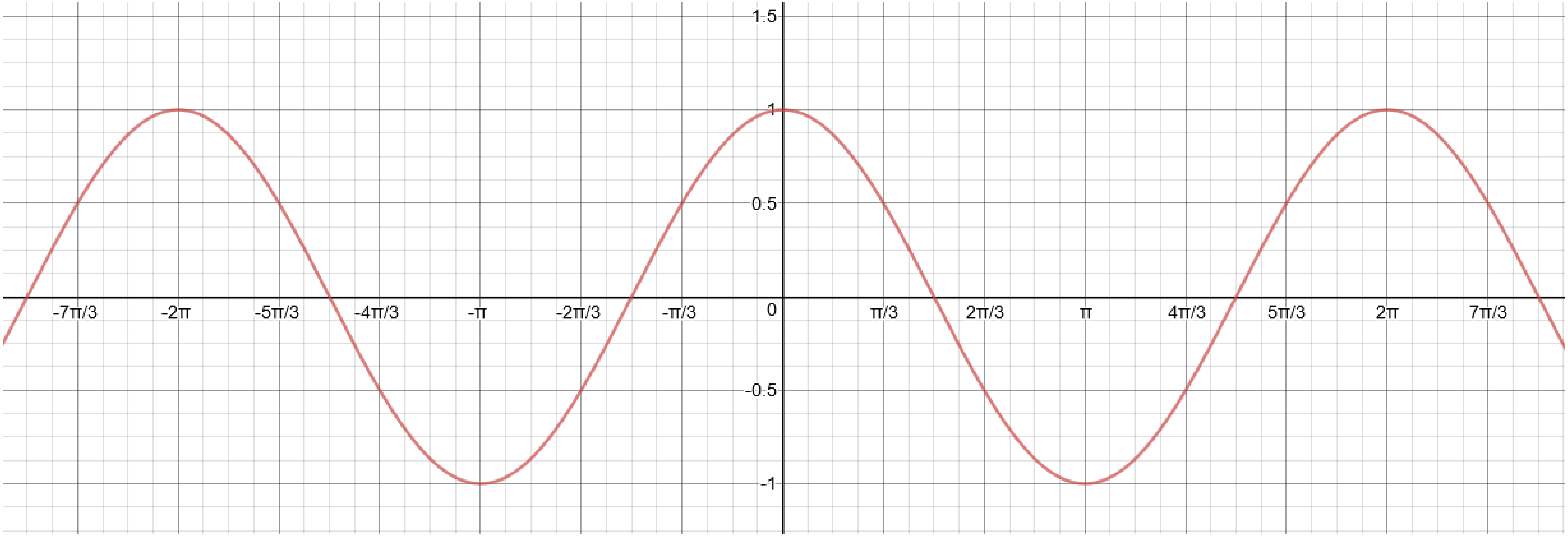
these are called periodic functions as they have a constant period of repetition.

Definition: $r = \sin(\theta)$



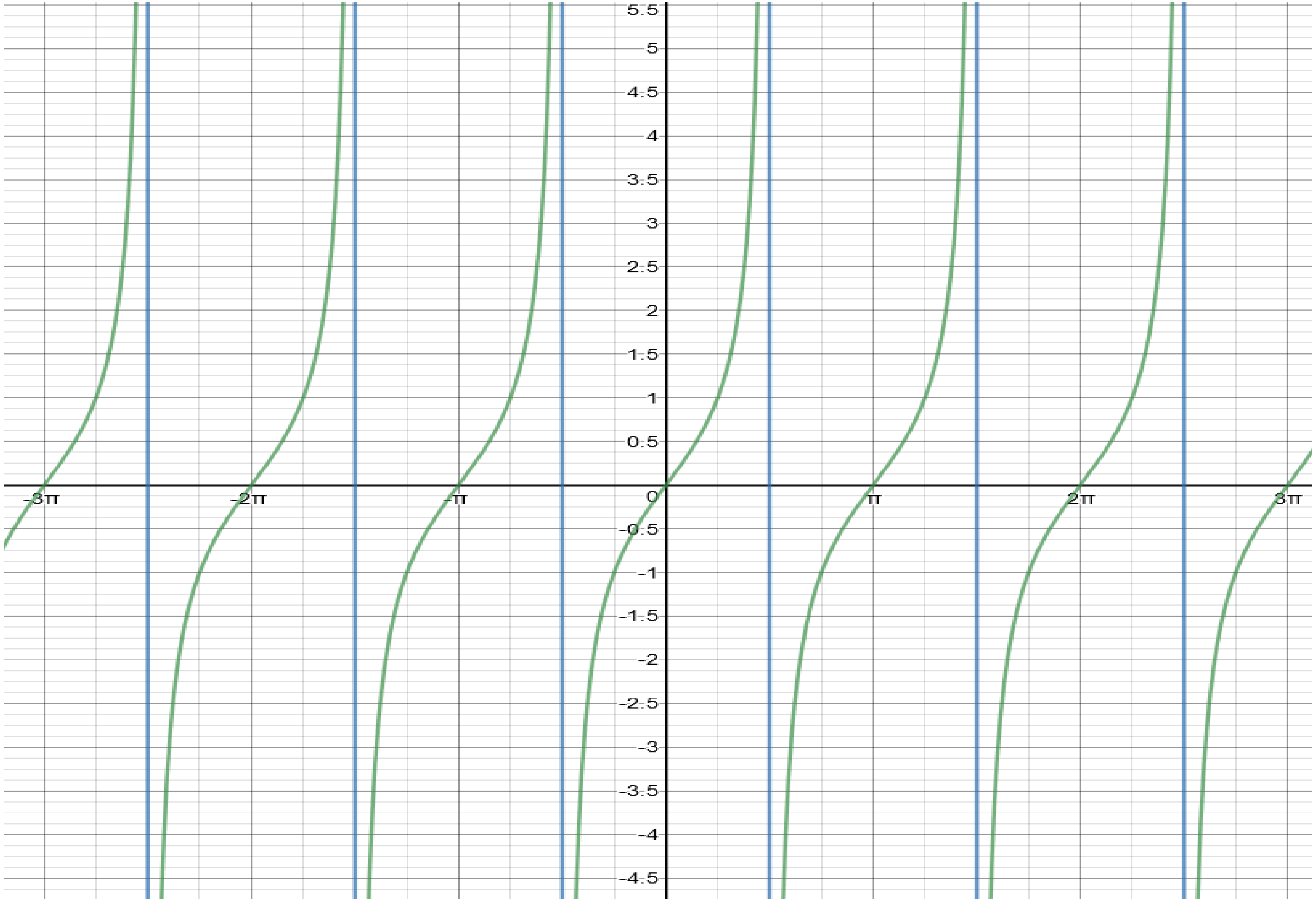
Domain: $(-\infty, \infty)$
Range:[-1,1]
Period:2π
Odd function

Definition: $r = \cos(\theta)$



Domain: $(-\infty, \infty)$
Range:[-1,1]
Period:2π
Even function

Definition: $r = \tan(\theta)$



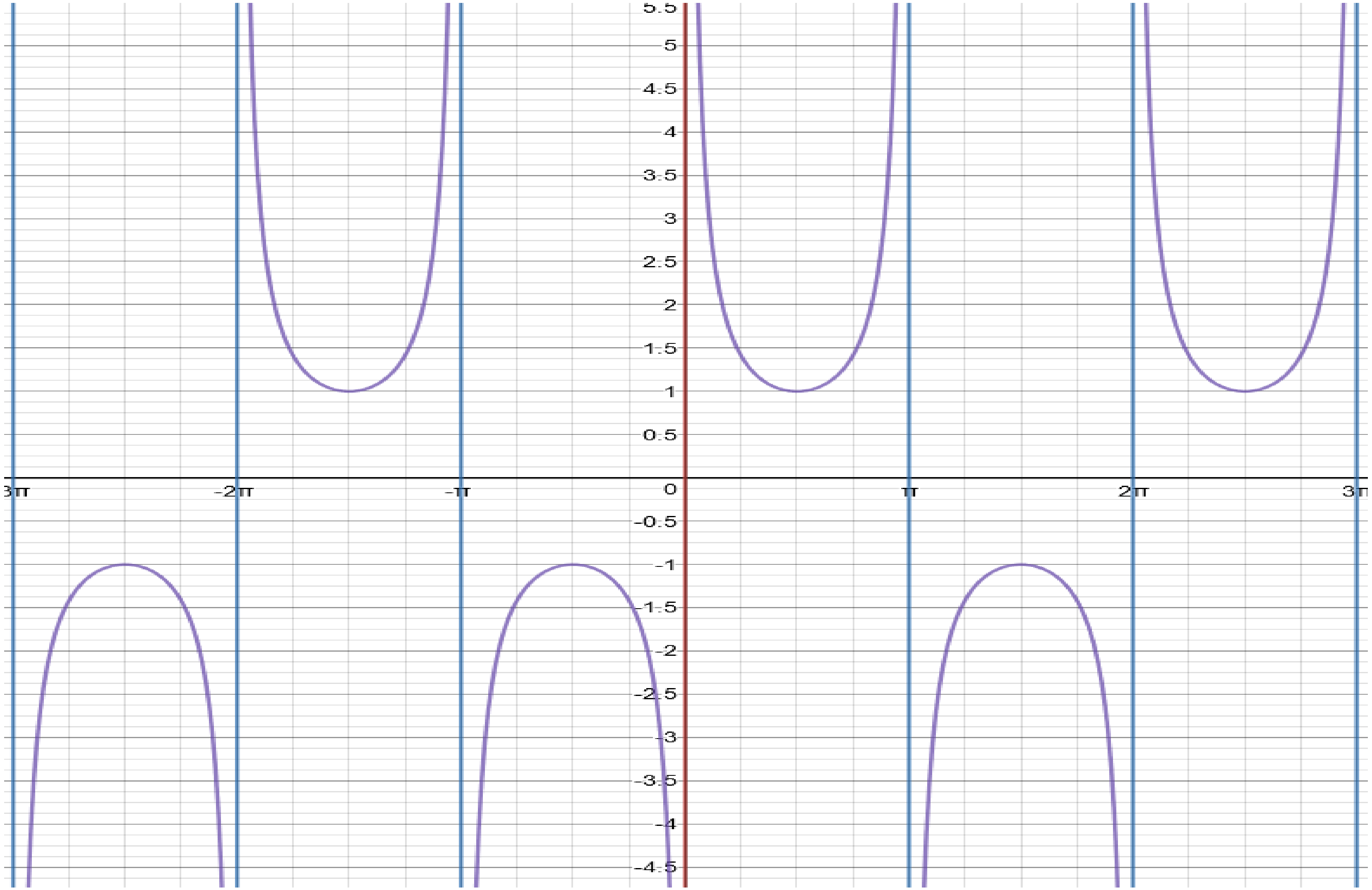
Domain: $\neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

Range: $(-\infty, \infty)$

Period: π

Odd function

Definition: $r = \csc(\theta)$



Domain: $\neq 0, \pm\boldsymbol{\pi}, \pm 2\boldsymbol{\pi}$
Range: $(-\infty, -1] \cup [1,, \infty)$
Period: 2π
Odd function

Definition: $r = \sec(\theta)$

Domain:$\neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

Range:$(-\infty, -1] \cup [1, , \infty)$

Period: 2π

Even function

Definition: $r = \cot(\theta)$

Domain: $\neq 0, \pm\boldsymbol{\pi}, \pm 2\boldsymbol{\pi}$
Range: $(-\infty, \infty)$
Period: π
Odd function

Examples: Graphing Trigonometric Functions

Example 4:

Graph the function $g(x) = 2\sin(2(x+\pi)) - 1$

Solution:

Parent function $f(x) = \sin(x)$

$g(x) = 2[f(2[x+\pi])] - 1$

$a = 2, b = 2, c = \pi, d = -1$

We generate our table of values for the parent function:

x	$f(x)$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

We then apply the transformations:

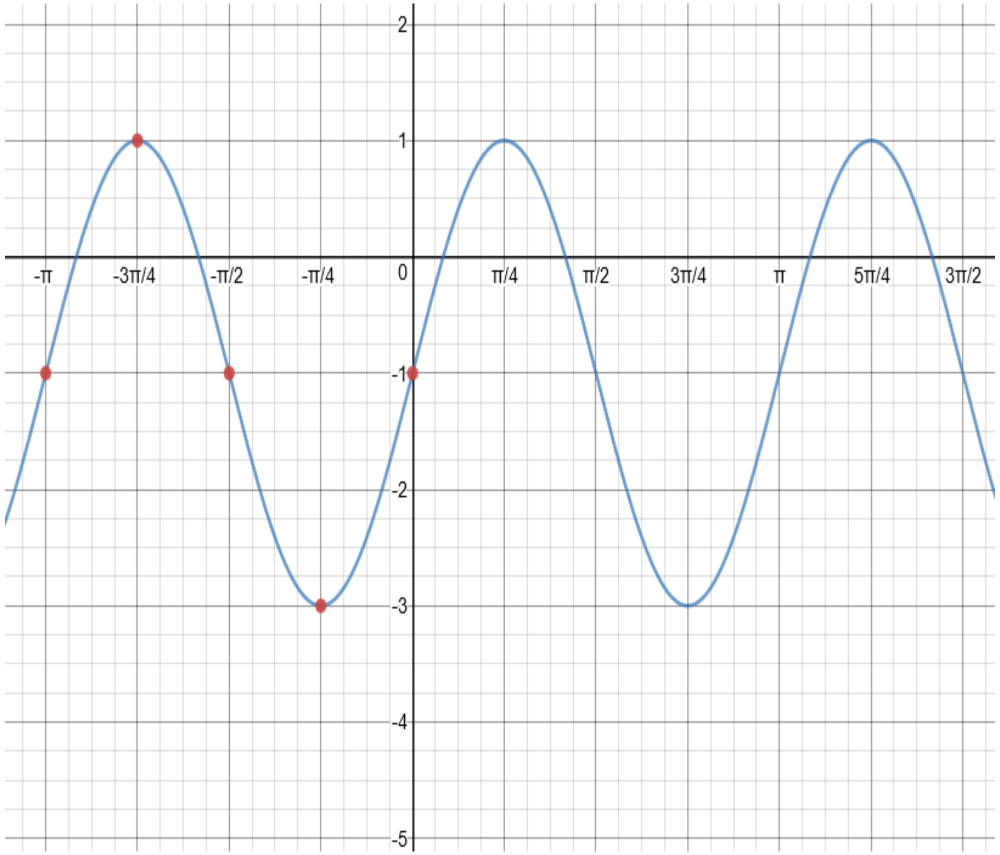
New x values:

$\frac{x}{b} - c \rightarrow \frac{x}{2} - \pi$

New y values:

$(y \times a) + d \rightarrow 2y - 1$

New x	$g(x)$
$\frac{0}{2} - \pi = -\pi$	$2(0) - 1 = -1$
$\frac{\pi/2}{2} - \pi = -\frac{3\pi}{4}$	$2(1) - 1 = 1$
$\frac{\pi}{2} - \pi = -\frac{\pi}{2}$	$2(0) - 1 = -1$
$\frac{3\pi/2}{2} - \pi = -\frac{\pi}{4}$	$2(-1) - 1 = -3$
$\frac{2\pi}{2} - \pi = 0$	$2(0) - 1 = 1$



Strategy: Solving Trigonometric Equations using the Unit Circle (when solving for the ratio)

How To Use it:

1) Identify the angle θ
2) Add (or subtract) 2π to get θ between 0 and 2π
3) Plot the angle on the unit circle.
4) Identify the point.
5) Use the ratio to evaluate the expression $(\sin(\theta) = y, \cos(\theta) = x, etc \ldots)$

When To Use it:

When evaluating a trig function when θ is known and is related to an angle on the unit circle.

Why this works?

This is simply creating an organized plan to use the definition of our trig ratios.

Note that we can add or subtract 2π increments as the unit circle repeats itself after each full circle.

Example 5:

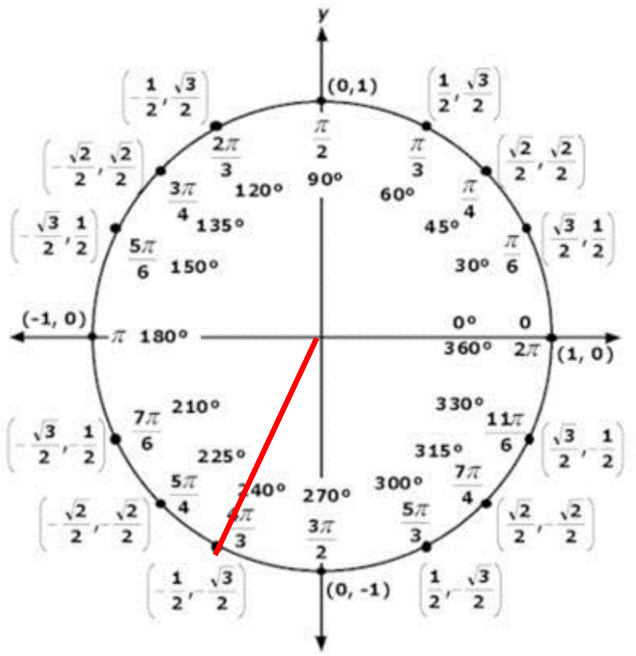
Determine $\tan\left(-\frac{2\pi}{3}\right)$

Solution:

1) $\theta = -\frac{2\pi}{3}$

2) We add 2π to get the angle within 0 and 2π. This gives us the corresponding angle $\frac{4\pi}{3}$

3) We plot the angle on the unit circle:



4) We see that $x = -\frac{1}{2}$ and $y = -\frac{\sqrt{3}}{2}$

5) $\tan\left(-\frac{2\pi}{3}\right) = \frac{y}{x} = -\frac{\sqrt{3}}{2} \div -\frac{1}{2} = \sqrt{3}$

Strategy: Solving Trigonometric Equations using the Unit Circle (when solving for the angle)

How To Use it:

1) Isolate the trig function on one side and have the ratio reduced on the other side.
2) Use the ratio to determine the location(s) of the point(s) on the unit circle ($\sin(\theta) = y, \cos(\theta) = x, etc \ldots$)
3) Determine the angle and solve for initial θ at the point(s).
4) Identify the period of the trig function:

   i) Sin, Cos, Sec, and Csc:

   $$per = \frac{2\pi}{b}$$

   ii) Tan and Cot

   $$per = \frac{\pi}{b}$$

5) The solution will be θ will be

   $$\theta = initial\ theta + (per)k, k \in \boldsymbol{Z}$$

When To Use it:

When solving for θ with a trigonometric function.

Why this works?

This is simply creating an organized plan to use the definition of our trig ratios.

Note that we have $+(per)k, k \in Z$ as the function repeats itself every period (either forward or backwards)

Example 6:

Solve $3\cot(2\theta) = \sqrt{3}$.

Solution:

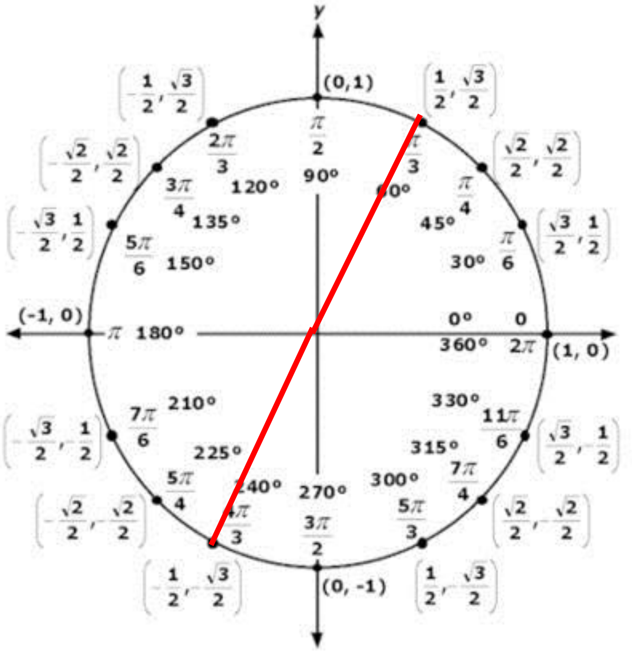
1) $\cot(2\theta) = \frac{\sqrt{3}}{3}$

2) On the unit circle, we see that $\cot(2\theta) = \frac{x}{y}$. We note that we have $\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$, so we are looking for an x with a 1 and a y with a $\sqrt{3}$ (and the division comes out as positive, so either both are positive or both are negative).

3) This means we have $2\theta = \frac{\pi}{3}$ or $2\theta = \frac{4\pi}{3}$ which solves to become $\theta = \frac{\pi}{6}$ or $\frac{2\pi}{3}$

4) Since our b value is 2 and our function is cot, this means that our period is $\frac{\pi}{b} = \frac{\pi}{2}$.

5) $\theta = \frac{\pi}{6} + \frac{\pi}{2}(k)$, $k \in \boldsymbol{Z}$ $\qquad\qquad$ $\theta = \frac{2\pi}{3} + \frac{\pi}{2}(k)$, $k \in \boldsymbol{Z}$



Formula: Trigonometric Identities

A <u>trigonometric identity</u> is an equation that involves a trigonometric function that always holds true no matter what input we place inside of the function.

Formula(s):	**When To Use it:**	**Proof of Formula(s)**
$\sin^2(\theta) + \cos^2(\theta) = 1$ $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$ $\cos(2\theta) = 2\cos^2(\theta) - \mathbf{1}$ $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ $\sec(\theta) = \frac{1}{\cos(\theta)}$ $\csc(\theta) = \frac{1}{\sin(\theta)}$ $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$	**1) When evaluating exact values of trigonometric functions** **2) Solving trigonometric equations.**	**To see proofs of these formulas, feel free to see:** **<u>Proofs of Trig Identities</u>**

Examples: Using Trigonometric Identities

Example 7:

Using trigonometric identities, find the exact value of $\sin\left(\frac{7\pi}{12}\right)$.

Solution:

We note that there is no special triangle that includes $/12$, but if we can use one of our identities, we can perhaps break the fraction up into pieces that we know how to find using our unit circle ($\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ or $\frac{\pi}{6}$). In this case we ask, how can we get $\frac{7\pi}{12} = ?\frac{\pi}{4} + ?\frac{\pi}{3}$

We would note that $7\pi = 3\pi + 4\pi$ and so

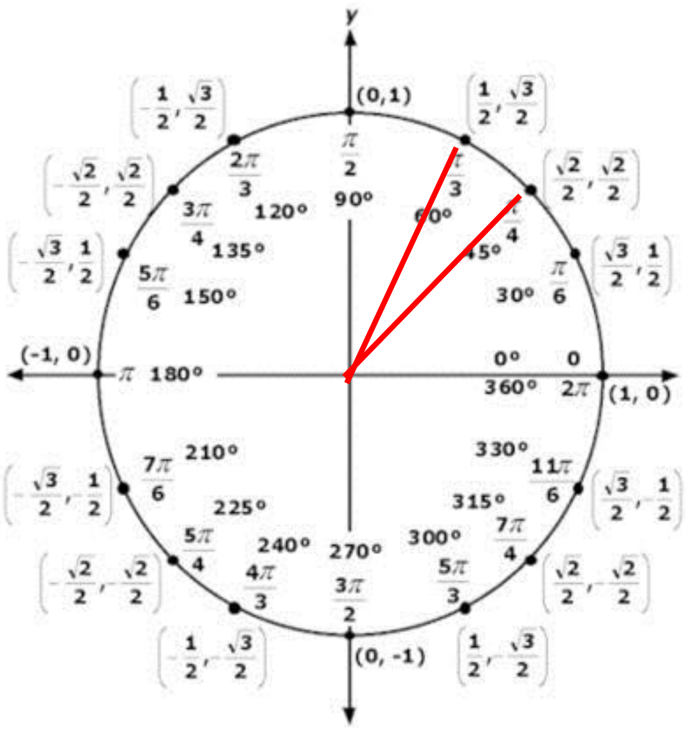
$$\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12}$$

$$\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$$

This allows us to use our sum formula identity:

$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\\ &= \sin(A)\cos(B) + \sin(B)\cos(A)\\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)\end{aligned}$$

Using our unit circle, we see:



Which gives

$$\begin{aligned}&= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Examples: Using Trigonometric Identities

Example 8:

Solve the following equation using identities: $\cos(\theta) - \cos(2\theta) = 0$

Solution:

When solving trigonometric equations with different arguments we want to get the arguments the same as a first step. In this case we can use the identity: $\cos(2\theta) = 2\cos^2(\theta) - 1$ which gives:

$\cos(\theta) - (2\cos^2(\theta) - 1) = 0$

$2\cos^2(\theta) - \cos(\theta) - 1 = 0$

$(2\cos(\theta) + 1)(\cos(\theta) - 1) = 0$

$\therefore \cos(\theta) = 1$ or $\cos(\theta) = -\frac{1}{2}$

We solve each expression to get:

For $\cos(\theta) = 1 = x$

$x = 1$

Period = $\frac{2\pi}{1} = 2\pi$

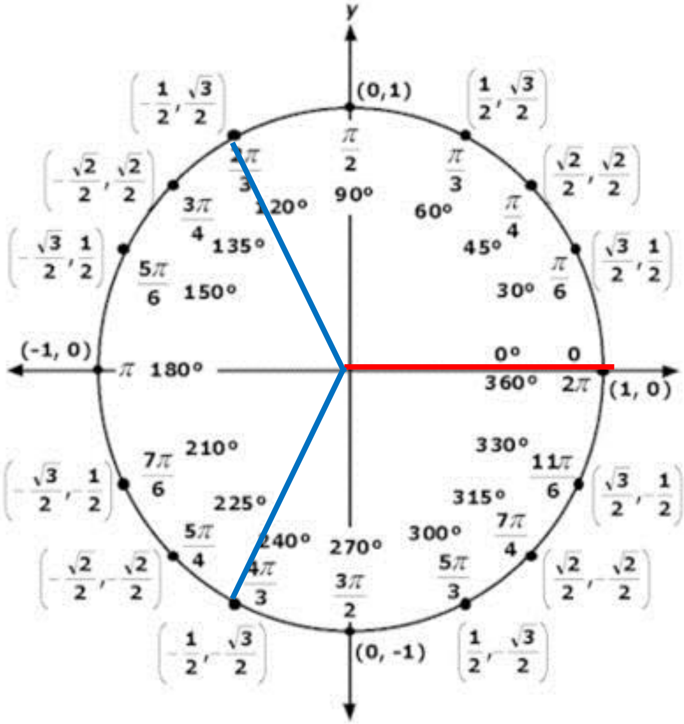
$\therefore \theta = 0 + 2\pi k, k \in \mathbf{Z}$

For $\cos(\theta) = -1/2 = x$

$x = -\frac{1}{2}$

Period = $\frac{2\pi}{1} = 2\pi$

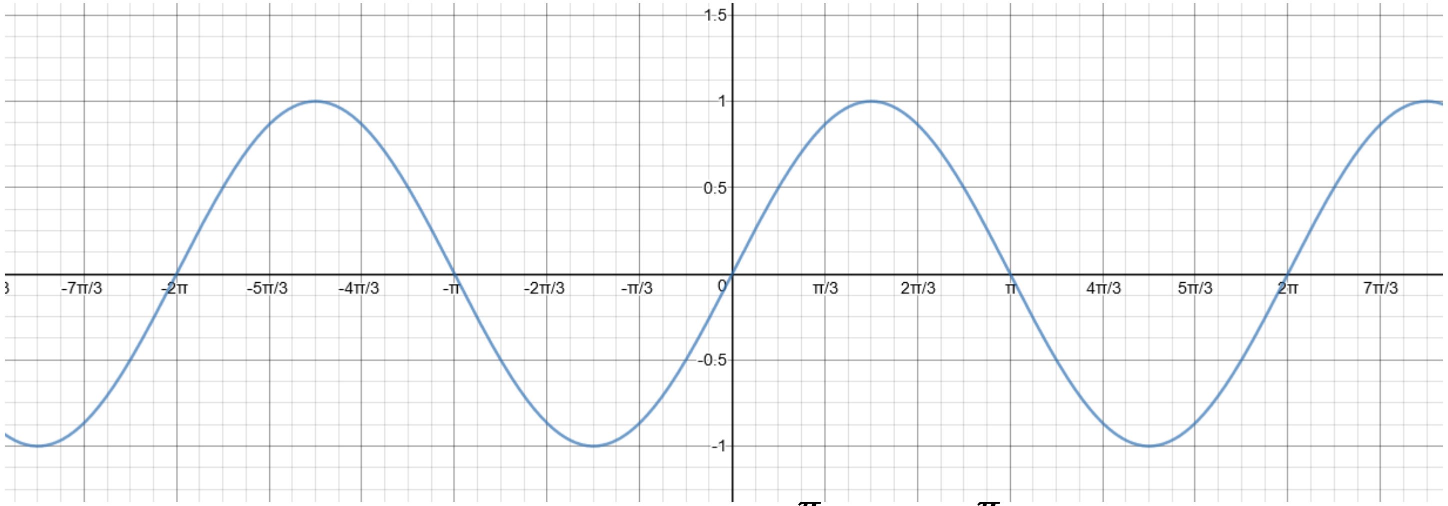
$\therefore \theta = \frac{2\pi}{3} + 2\pi k, k \in \mathbf{Z}$ or $\theta = \frac{4\pi}{3} + 2\pi k, k \in \mathbf{Z}$



Definition: Inverse Trigonometric Function

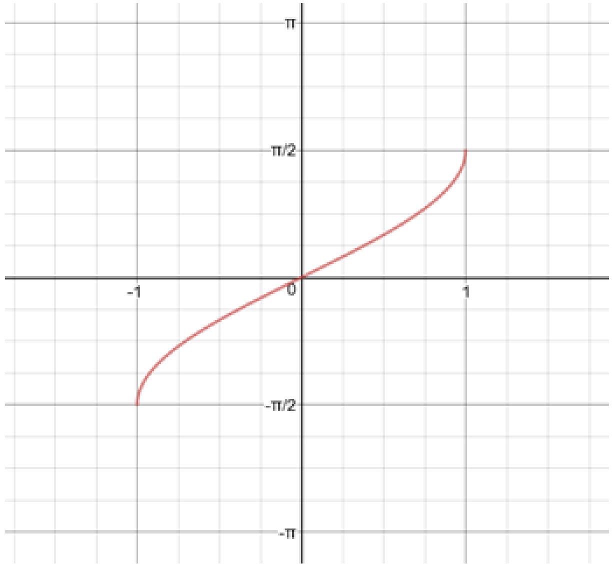
We define $Arcsin(r)$, $Arccos(r)$, and $Arctan(r)$ to be the inverse functions of $Sin(\theta)$, $Cos(\theta)$, and $Tan(\theta)$. Since we require these new relations to be functions, it imposes a domain restriction on the original function. What piece of the function will pass the "horizonal line test"?

$r = \sin(\theta)$



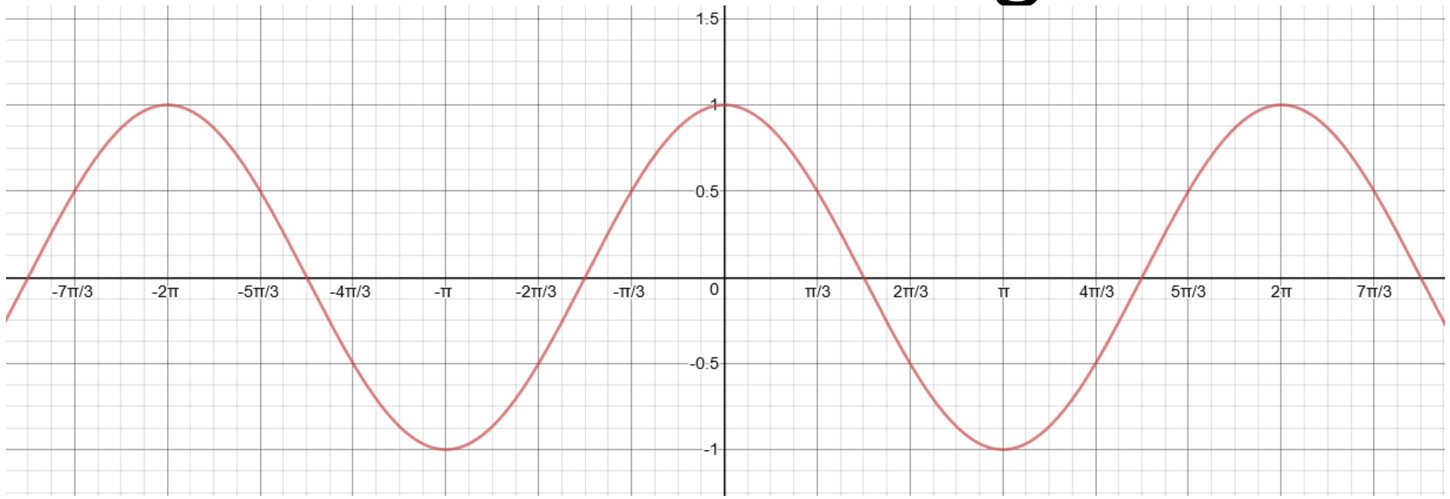
$\therefore$ $\boldsymbol{Arcsin(r)}$ enforces a hidden domain restriction of $\boldsymbol{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$.

This gives us the following graph for $\theta = Arcsin(r)$:



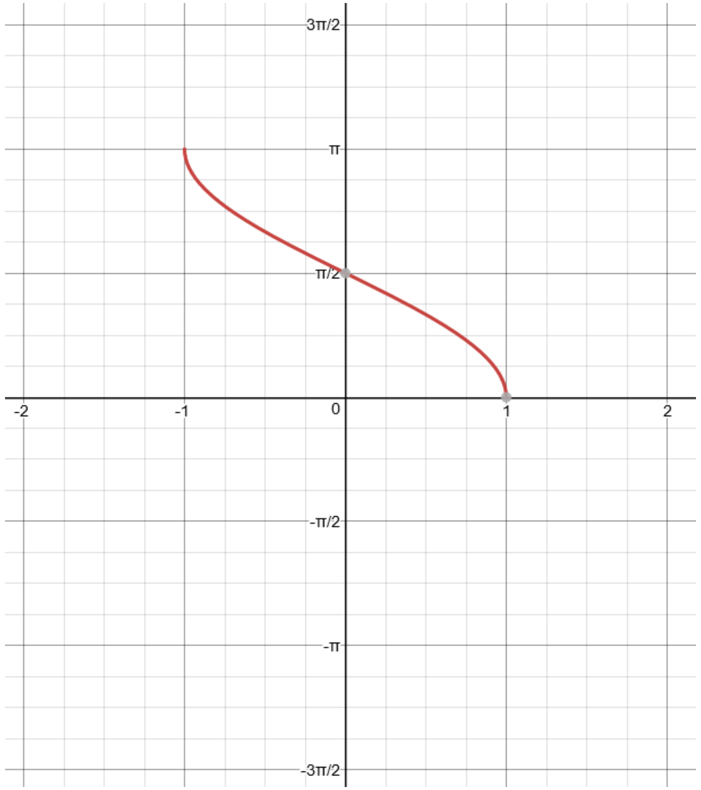
Definition: Inverse Trigonometric Function

$r = \cos(\theta)$



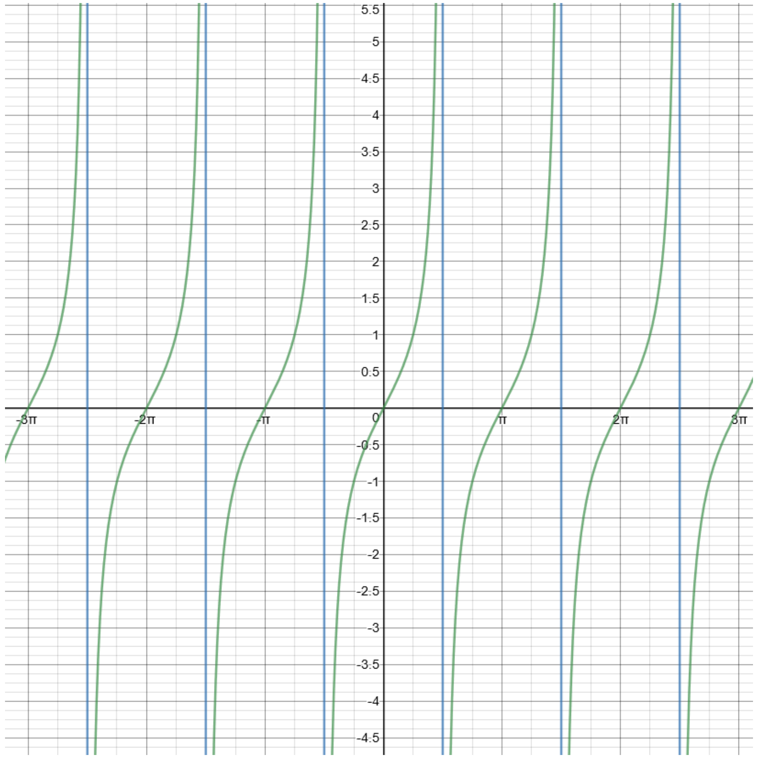
∴ $\boldsymbol{Arccos(r)}$ enforces a hidden domain restriction of $\boldsymbol{0 \leq \theta \leq \pi}$

This gives us the following graph for $\theta = Arccos(r)$:



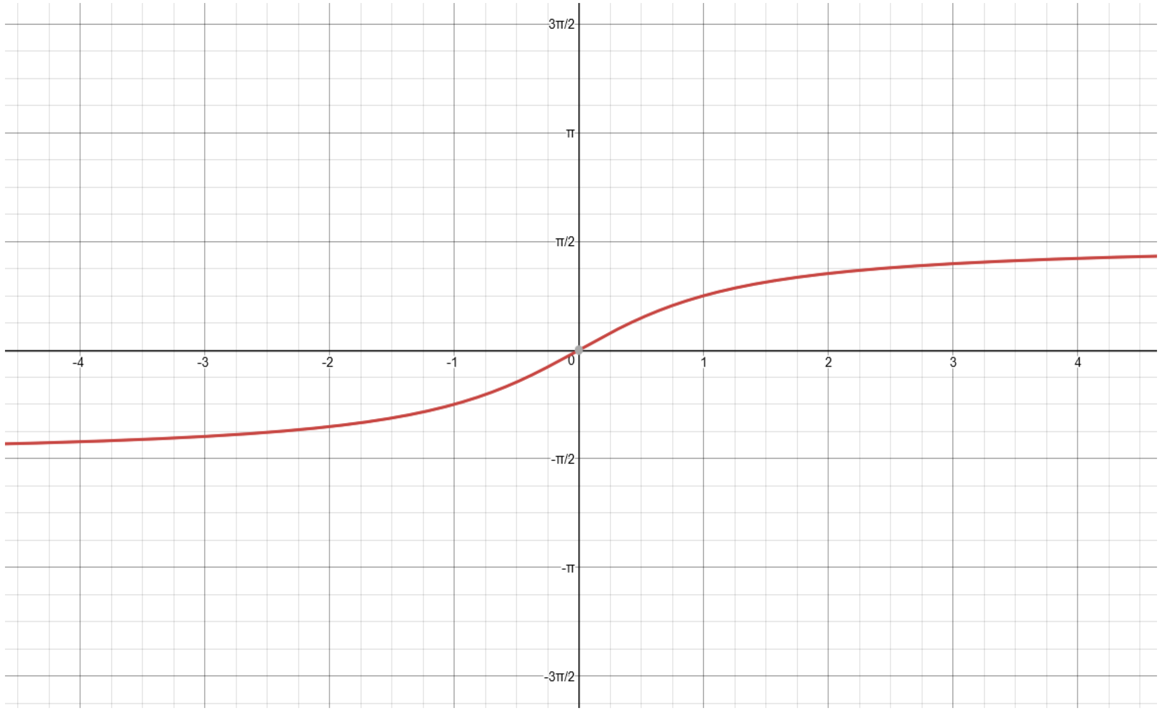
Definition: Inverse Trigonometric Function

$r = \tan(\theta)$



$\therefore$ $\boldsymbol{Arctan(r)}$ **enforces a hidden domain restriction of** $\boldsymbol{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}$

This gives us the following graph for $\theta = Arctan(r)$:



Examples: Inverse Trigonometric Functions

Example 6:

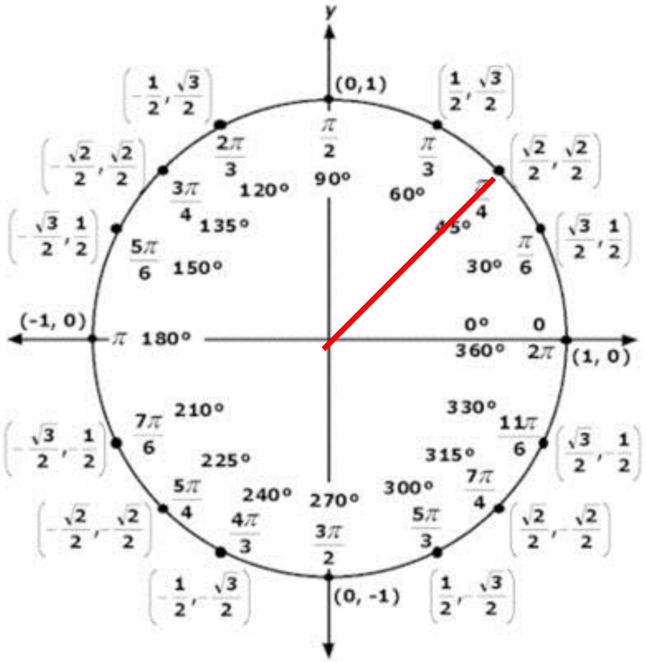
Simplify the following $\tan\left(\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$

Solution:

To solve this, we work on the inner function first: Let $\theta = \arcsin\left(\frac{\sqrt{2}}{2}\right)$

Since we are using an **inverse trig function**, we have a hidden domain restriction of $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ (the arcsin restriction).

Thus applying sin to both sides yields $\sin(\theta) = \sin\left(\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$ which simplifies to $\sin(\theta) = \frac{\sqrt{2}}{2} = y$. Looking to the unit circle, we get:



This gives us $\theta = \frac{\pi}{4}$. (We cannot select $\theta = \frac{3\pi}{4}$ as it is outside of the hidden domain restriction).

We simplify our result to get:

$$
\begin{aligned}
\tan\left(\arcsin\left(\frac{\sqrt{2}}{2}\right)\right) &= \tan(\theta) \\
&= \tan\left(\frac{\pi}{4}\right) \\
&= \frac{y}{x} \\
&= \frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} \\
&= 1
\end{aligned}
$$

Examples: Inverse Trigonometric Functions

Example 7:

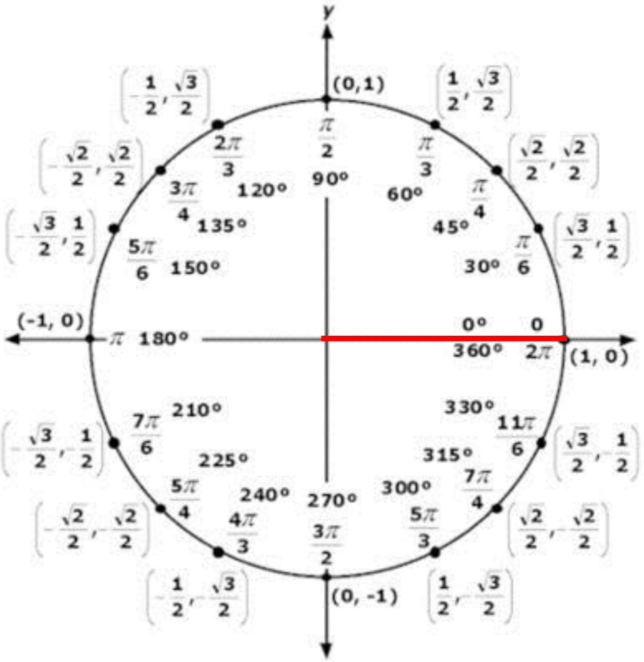
Simplify the following $\arccos(\cos(-2\pi))$

Solution:

This question is actually very misleading. One would think we can simply cancel the cos and the arccos, but -2π is not in our domain restriction for cos (which is $0 \leq \theta \leq \pi$). Thus we solve this more carefully:

We first note that we have a negative argument for cos, but cos repeats itself every 2π which means we can add 2π to the argument to get a simpler answer (and/or to use the unit circle). This means we get:

$$\begin{aligned} \arccos(\cos(-2\pi)) &= \arccos(\cos(0)) \\ &= \arccos(1) \end{aligned}$$



Let $\theta = \arccos(1)$ then $\cos(\theta) = 1 = x$. Here we can use the unit circle to solve the question:
This gives us our result of $\theta = 0$ thus our solution is $\arccos(\cos(-2\pi)) = \arccos(1) = 0$.

Note: We could have gotten there a bit faster by understanding that once we adjusted the argument to become 0, since this is in the domain restriction imposed by arccos ($0 \leq \theta \leq \pi$) we can conclude that $\arccos(\cos(0)) = 0$ (i.e they cancel).

Examples: Even and Odd Functions

Example :

Determine which of the following are even, odd, or neither:

a) $f(x) = x^4 + \cos(x) + 1$
b) $g(x) = 3x - x^3$
c) $h(x) = x \sin(x)$
d) $i(x) = \cos(x) + \sin(x)$

Solution:

Part a)

$f(-x) = (-x)^4 + \cos(-x) + 1$
$= x^4 + \cos(-x) + 1$ (Simplifying)
$= x^4 + \cos(x) + 1$ (Since cos is even we have $\cos(-x) = \cos(x)$)

$\therefore$ We have $f(-x) = f(x)$ and so the function is even.

Part b)

$g(-x) = 3(-x) - (-x)^3$
$= -3x + x^3$ (Simplifying)

$\therefore$ We can see that $g(-x) \neq g(x)$ but if we multiply both sides by -1 we get: $-g(-x) = 3x - x^3 = g(x)$

$\therefore$ We see that the function is odd.

Examples: Even and Odd Functions

Example Continued:

Determine which of the following are even, odd, or neither:

a) $f(x) = x^4 + \cos(x) + 1$
b) $g(x) = 3x - x^3$
c) $h(x) = x\sin(x)$
d) $i(x) = \cos(x) + \sin(x)$

Solution:

Part c)

$h(-x) = (-x)\sin(-x)$
$= (-x)(-\sin(x))$ (Since sin is odd, we know that $-\sin(-x) = \sin(x)$ or $-\sin(x) = \sin(-x)$)
$= x\sin(x)$

$\therefore$ We have $h(-x) = h(x)$ and so the function is even.

Part d)

$i(-x) = \cos(-x) + \sin(-x)$
$= \cos(x) - \sin(x)$ (Since cos is even and sin is odd)

$\therefore$ We can see that $i(-x) \neq i(x)$ but if we multiply both sides by -1 we get: $-i(-x) = \sin(x) - \cos(x) \neq i(x)$

$\therefore$ We see that the function is neither even nor odd.